JOURNAL OF APPROXIMATION THEORY **30**, 29–36 (1980)

Remarks on Some Conjectures of G. G. Lorentz

E. B. SAFF*

Department of Mathematics, University of South Florida, Tampa, Florida 33620

AND

R. S. VARGA†

Department of Mathematics, Kent State University, Kent, Ohio 44242

Communicated by Oved Shisha

Received February 15, 1979

Let f be any continuous real-valued function on the interval $[-1, 1]$ (i.e., $f \in C[-1, 1]$), and let $p_n^* = p_n^*(\cdot; f)$ denote, for each $n \geqslant 0$, its unique best uniform approximation from π_n on $[-1, 1]$, where π_n denotes the collection of all real polynomials of degree at most n. Then, it is well known (cf. [2, p. 34, Exercise 3]) that the assumption that f is odd implies that each p_n^* is odd, whence $p_n^*(0) = 0$ for every $n \geqslant 0$. Recently, Lorentz [3, 4] conjectured that the converse is also true, i.e.,

LORENTZ CONJECTURE 1. *If, for any $f \in C[-1, 1]$,*

$$p_n^*(0; f) = 0, \qquad \textit{for all} \quad n \geqslant 0, \tag{1}$$

then f is odd.

In addition, Lorentz [4] has made the following related conjectures:

LORENTZ CONJECTURE 2. *If, for any $f \in C[-1, 1]$, there is an $\alpha \neq 0$ in $[-1, 1]$ for which*

$$p_n^*(\alpha; f) = 0 \qquad \textit{for all} \quad n \geqslant 0, \tag{2}$$

then $f \equiv 0$.

* Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2688 and by a John Simon Guggenheim Memorial Fellowship.

† Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2729, and by the Department of Energy under Grant DOE EY-76-02-2075.

0021-9045/80/090029–08$02.00/0

LORENTZ CONJECTURE 3. *If, for any $f \in C[-1, 1]$,*

$$p_{2k+1}^*(x; f) = p_{2k+2}^*(x; f) \qquad \text{for all} \quad k \geqslant 0, \tag{3}$$

then f is odd.

LORENTZ CONJECTURE 4. *If, for any $f \in C[-1, 1]$,*

$$p_{2k}^*(x; f) = p_{2k+1}^*(x; f) \qquad \text{for all} \quad k \geqslant 0, \tag{4}$$

then f is even.

The object of this note is to give partial answers to all of the above conjectures.

To begin, given any $f \in C[-1, 1]$ and any nonnegative integer n, set

$$\epsilon_n(x) := f(x) - p_n^*(x), \qquad \forall x \in [-1, 1], \tag{5}$$

and put

$$E_n(x) := \epsilon_n(x) + \epsilon_n(-x) = [f(x) + f(-x)] - [p_n^*(x) + p_n^*(-x)], \tag{6}$$

so that E_n is an *even* function on $[-1, 1]$ for all $n \geqslant 0$. It is well known (cf. [2, p. 30]) that there exist at least $n + 2$ distinct alternation points $\xi\{\xi_j^{(n)}\}_{j=1}^{n+2}$ such that

$$-1 \leqslant \xi_1^{(n)} < \xi_2^{(n)} < \cdots < \xi_{n+2}^{(n)} \leqslant 1, \tag{7}$$

and

$$\| f - p_n^* \|_{L_\infty[-1,1]} = \lambda(-1)^j \, \epsilon_n(\xi_j^{(n)}), \qquad 1 \leqslant j \leqslant n + 2, \tag{8}$$

where $\lambda = 1$ or $\lambda = -1$.

Noting that $| \epsilon_n(-\xi_j^{(n)})| \leqslant \| f - p_n^* \|_{L_\infty[-1,1]}$, we have from (8) that

$$\begin{aligned} &\lambda(-1)^j [\epsilon_n(\xi_j^{(n)}) + \epsilon_n(-\xi_j^{(n)})] \\ &\qquad = \| f - p_n^* \|_{L_\infty[-1,1]} + \lambda(-1)^j \, \epsilon_n(-\xi_j^{(n)}) \geqslant 0, \end{aligned}$$

and hence, for the function $E_n(x)$ in (6), there holds

$$\lambda(-1)^j \, E_n(\xi_j^{(n)}) \geqslant 0 \qquad \forall 1 \leqslant j \leqslant n + 2, \qquad \forall n \geqslant 0. \tag{9}$$

Now, this oscillatory behavior of E_n implies, from the continuity of E_n, the existence of zeros of E_n in $[-1, 1]$. The following easily verified lemma gives a more precise form of this observation.

LEMMA 1. *Let $f \in C[-1, 1]$ satisfy*

$$[f(x) + f(-x)] \in C^1[-1, 1]. \tag{10}$$

Then, for each $n \geqslant 0$, E_n has (counting multiplicities) at least $n + 1$ zeros in $[-1, 1]$, where each zero of E_n is counted as having a multiplicity of order at most 2, and where any zero of E_n counted as having a multiplicity of order 2 is a $\xi_n^{(j)}$ for some j, $1 \leqslant j \leqslant n + 2$.

Concerning Lorentz Conjecture 1, we now assume that $f \in C[-1, 1]$ is such that

$$p_{2k+1}^*(0; f) = 0, \qquad \forall k \geqslant 0, \tag{11}$$

which is a weaker assumption than that made in (1). Evidently, since the sequence $\{p_n^*(\cdot; f)\}_{n=0}^{\infty}$ converges to f on $[-1, 1]$, the assumption of (11) implies that

$$f(0) = 0, \tag{12}$$

whence (cf. (6))

$$E_{2k+1}(0) = 0, \qquad \forall k \geqslant 0. \tag{13}$$

Thus, in the case when $\| f - p_{2k+1}^* \|_{L_\infty[-1,1]} > 0$ (the remaining case being trivial), it follows from (11) and (12) that no $\xi_j^{(2k+1)}$ can equal zero. But as $E_{2k-1}(x)$ is even with $E_{2k+1}(0) = 0$, its (at least) double order zero at $x = 0$ is counted only once in Lemma 1. Hence, E_{2k+1} has at least $2k + 3$ zeros in $[-1, 1]$ and thus, because of evenness, there must be at least $2k + 4$ zeros in $[-1, 1]$. This is stated as

LEMMA 2. *Let $f \in C[-1, 1]$ be such that (10) and (11) hold. Then, for each $k \geqslant 0$, the function E_{2k+1} has (counting multiplicities) at least $2k + 4$ zeros in $[-1, 1]$, where each zero is counted as having a multiplicity of order at most 2.*

Next, because of evenness considerations, we can write

$$p_{2k+1}^*(x; f) + p_{2k+1}^*(-x; f) =: s_k(x^2), \qquad \forall k \geqslant 0, \tag{14}$$

where $s_k \in \pi_k$. Similarly, we can write

$$f(x) + f(-x) =: F(x^2), \tag{15}$$

and we then set

$$R_k(t) := F(t) - s_k(t), \qquad t \in [0, 1], \forall k \geqslant 0. \tag{16}$$

This brings us to our first main result, which establishes the partial validity of Lorentz Conjecture 1.

PROPOSITION 1. *Let $f \in C[-1, 1]$ satisfy the conditions:*

(i) $p_{2k+1}^{*}(0; f) = 0 \; \forall k \geqslant 0$, *and*

(ii) *the function $F(t)$ defined in* (15) *has an analytic extension $F(z)$ which is an entire function of exponential type τ with $0 \leqslant \tau < \pi/2$, i.e., (cf. Boas [1, p. 8]),*

$$\limsup_{r\to\infty} \frac{\ln M_F(r)}{r} = \tau < \frac{\pi}{2}, \qquad \textit{where} \qquad M_F(r) := \max\{|F(z)| : |z| = r\}. \tag{17}$$

Then, f is odd.

Proof. By condition (ii), the function $R_k(t)$ defined in (16) satisfies $R_k \in C^\infty[0, 1]$. Furthermore, interpreting the result of Lemma 2 for $R_k(t)$, it follows that $R_k(t)$ has at least $k + 2$ zeros in $[0, 1]$. Thus, by the generalized Rolle's Theorem, there exists a $\beta_{k+1} \in [0, 1]$ for which

$$R_k^{(k+1)}(\beta_{k+1}) = 0, \qquad \forall k \geqslant 0. \tag{18}$$

But since $s_k \in \pi_k$, (18) implies from (16) that

$$F^{(k+1)}(\beta_{k+1}) = 0, \qquad \forall k \geqslant 0. \tag{19}$$

Also, by condition (i), we have $F(0) = f(0) = 0$, and on setting $\beta_0 := 0$, (19) can be extended to

$$F^{(j)}(\beta_j) = 0, \qquad \text{with} \quad \beta_j \in [0, 1], \; \forall j \geqslant 0. \tag{20}$$

Next, defining $G(z) := F((z + 1)/2)$, it follows from (17) that G is entire of exponential type σ with $0 \leqslant \sigma < \pi/4$, and from (20) there exist γ_j's such that

$$G^{(j)}(\gamma_j) = 0, \qquad \text{with} \quad \gamma_j \in [-1, 1], \; \forall j \geqslant 0. \tag{21}$$

But using a classical result of Schoenberg [6], the above properties imply that $G(z) \equiv 0$, whence $F(z) \equiv 0$. Recalling the definition of F in (15), it follows that f must be odd. ∎

Remark 1. In Proposition 1, condition (ii) can be replaced by the stronger assumption

(ii)′ $f(x)$ has an analytic extension $f(z)$ which is an entire function of at most order 2 and type λ, where $0 \leqslant \lambda < \pi/2$.

Remark 2. As a special case of Proposition 1, we have that if f is any real polynomial function and if condition (i) holds, then f is odd.

As a simple application of Remark 2, consider any odd degree Zolotareff polynomial (cf. [5, p. 41])

$$Z_{2m+1}(x;\, \sigma) := x^{2m+1} + \sigma x^{2m} - q(x), \qquad m \geqslant 1,$$

where q is defined to be the best uniform approximation to $x^{2m+1} + \sigma x^{2m}$ on $[-1, 1]$ from π_{2m-1}. We prove that

$$Z_{2m+1}(0;\, \sigma) \neq 0 \qquad \text{for any} \quad \sigma \neq 0, \text{ any } m \geqslant 1.$$

Indeed, if we assume on the contrary that $Z_{2m+1}(0;\, \sigma) = 0$ for some $\sigma \neq 0$, then $p^*_{2k+1}(0;\, Z_{2m+1}) = Z_{2m+1}(0;\, \sigma) = 0$ for all $k \geqslant m$. Also, from the definition of the Zolotareff polynomials, it follows that $Z_{2m+1}(x;\, \sigma)$ has an alternation set consisting of at least $2m + 1$ distinct points in $[-1, 1]$, whence $p^*_{2k+1}(x;\, Z_{2m+1}) \equiv 0$ for all $0 \leqslant k \leqslant m - 1$. But then Remark 1 implies that $Z_{2m+1}(x;\, \sigma)$ is an odd function, which is absurd for $\sigma \neq 0$.

Remark 3. The assumption (i) in Proposition 1 cannot be weakened, i.e., no one condition of (i) can be deleted without destroying the conclusion of Proposition 1. Indeed, for each nonnegative integer m, there exists a polynomial function $f_m(x)$ such that

$$p^*_{2k+1}(0; f_m) = 0, \qquad \forall k \geqslant 0, \qquad k \neq m, \tag{22}$$

and such that f_m is *not* odd. To see this, let $T_n(x) = \cos(n \cos^{-1} x)$ denote the classical Chebyshev polynomial (of the first kind) of degree n and, for $m \geqslant 0$, set

$$Q_{m+1}(t) := T_{m+1}[(1 - \alpha_m)t + \alpha_m], \qquad \text{where} \quad \alpha_m := -\cos(\pi/2(m+1)). \tag{23}$$

We note that $Q_{m+1}(0) = 0$ and that $Q_{m+1}(t)$ has precisely $m + 1$ equioscillations on the interval $0 \leqslant t \leqslant 1$. Now, set

$$f_m(x) := Q_{m+1}(x^2), \qquad x \in [-1, 1]. \tag{24}$$

Since $f_m \in \pi_{2m+2}$, we have $p^*_{2k+1}(0; f_m) = f_m(0) = Q_{m+1}(0) = 0$, for all $k \geqslant m + 1$. Furthermore, from (24), the function $f_m(x)$ has an alternation set consisting of $2m + 1$ points in $[-1, 1]$, and hence $p^*_{2k+1}(x; f_m) \equiv 0$ for all $0 \leqslant k \leqslant m - 1$. Thus, (22) holds for all $k \neq m$; however, $f_m(x) (\not\equiv 0)$ is an *even* function of x.

Remark 4. If $f \in C[-1, 1]$, $f \not\equiv 0$, and if we assume *only* that condition (i) holds in Proposition 1, then we can deduce that f is certainly not even. Indeed, suppose $f(x)$ is an even function. Then, the polynomial $s_k(t)$ defined in (14) is the best uniform approximation from π_k to $F(t)$ on the interval $[0, 1]$ for all $k \geq 0$. But condition (i) implies that $s_k(0) = 0$ $\forall k \geq 0$, whence $s_{k+1}(t) - s_k(t)$ does not have all its zeros in the open interval $(0, 1)$. Using a lemma of Lorentz (cf. [3, p. 290]), this implies that $s_k(t) = s_{k-1}(t)$. As $s_0(t) \equiv 0$, then $s_k(t) \equiv 0$ for all $k \geq 0$ and hence $0 \equiv F(t) = f(\sqrt{t}) + f(-\sqrt{t}) = 2f(\sqrt{t})$, which contradicts the fact that $f \not\equiv 0$.

Concerning Lorentz Conjecture 2, consider the following *counterexample*. For each positive integer $m \geq 1$, define $\hat{f}_m(x) \in C[-1, 1]$ by means of

$$\hat{f}_m(x) := \sum_{j=m}^{\infty} b_j T_{3^j}(x), \qquad \text{where} \qquad b_j \geq 0 \ \forall j \geq 1, \quad 0 < \sum_{j=m}^{\infty} b_j < \infty, \tag{25}$$

where $T_n(x)$ denotes, as before, the Chebyshev polynomial (of the first kind) of degree n. From S. N. Bernstein, it is well known (cf. Lorentz [3, p. 290]) that the partial sums, $S_n(x; m)$, of $\hat{f}_m(x)$, defined by

$$\begin{aligned} S_n(x; m) &:= \sum_{j=m}^{n} b_j T_{3^j}(x), && \text{for} \quad n \geq m, \\ &:= 0, && \text{for} \quad 0 \leq n < m, \end{aligned} \tag{26}$$

are the polynomials of best uniform approximation to $\hat{f}_m$ on $[-1, 1]$, i.e.,

$$S_n(\cdot; m) = p^*_{3^n}(\cdot; \hat{f}_m) = p^*_{3^n+1}(\cdot; \hat{f}_m) = \cdots = p^*_{3^{n+1}-1}(\cdot; \hat{f}_m), \qquad \forall n \geq 0. \tag{27}$$

Now, let

$$\alpha_m := \sin(\pi/3^m), \qquad \forall m \geq 1. \tag{28}$$

Then, it is easily seen that $T_{3^j}(x)$ has α_m as a zero for each $j \geq m$, whence $S_n(\alpha_m; m) = 0$ for each $n \geq 0$. But, as $\hat{f}_m$ is not identically zero and as $\alpha_m \neq 0$, we see from (27) that $\hat{f}_m$ constitutes a counterexample to Lorentz Conjecture 2 for each $m \geq 1$. Note that $\alpha_m \downarrow 0$ as $m \to \infty$.

Of course, $T_{3^m}(x)$ itself, by the same reasoning, is also a counterexample. The reason that the more complicated $\hat{f}_m$ of (25) was considered is to show that if the restriction that "f is not a polynomial" is added to Lorentz Conjecture 2, this conjecture still remains false for certain choices of α.

We remark however that if $\alpha = 1$ or $\alpha = -1$ in Lorentz Conjecture 2, then indeed $f \equiv 0$. This follows, as in Remark 4, from the fact that $p^*_{n+1}(x; f) =$

$p_n^*(x; f)$ does not have all its zeros in the open interval $(-1, 1)$, for all $n \geq 0$. It still remains an open problem whether there is some particular value of $x \in (-1, 1)$ for which Lorentz Conjecture 2 is valid.

Turning now to Lorentz Conjecture 3, this conjecture is false as stated since $f(x) := x + 1$ has $p_0^*(x; f) \equiv 1$, and $p_n^*(x; f) = x + 1$ for all $n \geq 1$. Thus, $\{p_n^*(\cdot; f)\}_{n=0}^{\infty}$ satisfies Lorentz Conjecture 3 without f being odd. This suggests modifying this conjecture by adding the hypothesis $f(0) = 0$:

LORENTZ CONJECTURE 3′. *If, for any $f \in C[-1, 1]$ with $f(0) = 0$, we have*

$$p_{2k+1}^*(x; f) = p_{2k+2}^*(x; f), \qquad \forall k \geq 0, \tag{29}$$

then f is odd.

With the assumptions of (29), it follows that $\epsilon_{2k+1}(x) = \epsilon_{2k+2}(x)$ for each $k \geq 0$, so that $f - p_{2k+1}^*$ has at least $2k + 4$ distinct alternation points $\{\xi_j^{(2k+1)}\}_{j=1}^{2k+4}$ satisfying (7) and (8). Thus, Lemma 1 can be directly applied to deduce, as in Lemma 2, that $E_{2k+1}(x)$ has at least $2k + 4$ zeros (where each zero is counted as having a multiplicity of order at most 2), and moreover, the proof of Proposition 1 can be applied without change. Thus, we have the following partial affirmative result for the Lorentz Conjecture 3′:

PROPOSITION 2. *Let $f \in C[-1, 1]$ satisfy $f(0) = 0$ and $p_{2k+1}^*(x; f) = p_{2k+2}^*(x; f)$ for each $k \geq 0$. If the function $F(t)$ defined in* (15) *has an analytic extension $F(z)$ which satisfies* (17), *then f is odd.*

Turning finally to the Lorentz Conjecture 4, the basic approach we have previously used can, with minor modifications, be applied here as well. For brevity, we simply state the following partial affirmative result.

PROPOSITION 3. *Let $f \in C[-1, 1]$ satisfy $p_{2k}^*(x; f) = p_{2k+1}^*(x; f)$ for all $k \geq 0$. Define $\tilde{F}(t)$* by

$$f(x) - f(-x) := x\tilde{F}(x^2), \tag{30}$$

and assume that $\tilde{F}$ has an analytic extension $\tilde{F}(z)$ which satisfies (17). *Then, f is even.*

REFERENCES

1. R. P. BOAS, JR., "Entire Functions," Academic Press, New York, 1954.
2. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
3. G. G. LORENTZ, Approximation by incomplete polynomials (problems and results), "Padé and Rational Approximation" (E. B. Saff and R. S. Varga, Eds.), pp. 289–302, Academic Press, New York, 1977.

4. G. G. LORENTZ, New and unsolved problems, *in* "Linear Spaces and Approximation" (P. L. Butzer and B. Sz.-Nagy, Eds.), p. 665, Birkhäuser, Stuttgart, 1978.
5. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, New York, 1967.
6. I. J. SCHOENBERG, On the zeros of successive derivatives of integral functions, *Trans. Amer. Math. Soc.* **40** (1936), 12–23.